All varieties will be projective, $/\mathbb{C}$, normal.

$(X, \Delta)$ pair $X =$ variety

$\Delta$ $\mathbb{Q}$ divisor

sth $K_X + \Delta$ $\mathbb{Q}$ Cartier

(we do not assume $\Delta \geq 0$, and klt = sub klt
lc = sub lc)

Def An lc/klt-trivial fibration is

$(X, \Delta)$ pair, $f : X \to Y$ fibration

1* $(X, \Delta)$ is klt / lc over the generic point of $Y$

2* $\exists D$ $\mathbb{Q}$ Cartier on $Y$ sth $K_X + \Delta \sim_{\mathbb{Q}} f^* D$

3* there is $\pi : X' \to X$ log resol of $(X, \Delta)$ with the f property

set $\Xi = \pi^*(K_X + \Delta) - K_{X'}$

$\Xi = \Xi^{<1} + \Xi^{\geq 1}$ ← if f klt-triv is $(f \circ \pi)$-vertical

$\operatorname{rank} (f \circ \pi)_* \mathcal{O}_{X'}(\lceil -\Xi^{<1} \rceil) = 1$

NOT $f : (X, \Delta) \to Y$

TERMINOLOGY klt/lc trivial are all lc-trivial for Ambro.

Remarks on (3*) (I) $\lceil -(\Xi^{<1})\rceil$

$$-\Xi^{<1} = (-\Xi)^{>-1} \Rightarrow \lceil -\Xi^{<1}\rceil = \lceil (-\Xi)^{>-1}\rceil \geq 0$$
$$\qquad\qquad\qquad\qquad\qquad\qquad \| $$
$$\qquad\qquad\qquad\qquad\qquad\qquad \lceil -\Xi \rceil \geq 0$$

Condition (3*) says that "birationally" $\lceil -\Delta \rceil$ has no global sections $/Y$ other than $\lceil -\Delta \rceil$.

(II) the definition does not depend on the choice of $\pi$, and we can replace "there is" with "for all".

(III) (3*) is verified if $\Delta \geq 0$ over the generic point of $Y$, or if $f: X \xrightarrow{\mu} \bar{X} \longrightarrow Y$, $\mu$ birational, $\mu_* \Delta \geq 0$.

Def $f:(X,\Delta) \to Y$ lc-trivial, $P \subseteq Y$ prime divisor.

$$\gamma_P = \sup\{t \in \mathbb{R}_{\geq 0} \mid (X, \Delta + tf^*P) \text{ is lc over the generic point of } P\}$$

Rk (1) definition of $\gamma_P$ if $P$ not cartier we restrict to $Y_{sm}$, then $f^*P$ is defined

(2) lc over the gen pt of $P$

$\exists \pi: X' \longrightarrow X$ log res of $(X, \tilde{\Delta})$ sth

$\forall E \subseteq X'$ s.t.h. $f(\pi(E)) = P$ $\quad a(E, X, \bar{\Delta}) \geq -1$

**Def** The discriminant is $B_Y = \sum\limits_{\substack{P \subseteq Y \\ \text{codim } 1}} (1-\gamma_P) \cdot P$

The moduli part is $M_Y = D - K_Y - B_Y$

$\hat{}$ s.t.h. $K_X + \Delta \sim_{\mathbb{Q}} f^* D$

Comments

- the def of discriminant is due to Kawamata
- the discriminant depends only on $f$
  the mod part depends on $D$ as well
  it is defined as a $\mathbb{Q}$-linear equiv class.
- Motto $\quad$ Supp $B_Y$ = singular locus of $f: (X, \Delta) \to Y$
  $= \bigcup \{P \mid (X, \Delta + f^* P) \text{ is not lc}\}$

EASY PROPERTIES OF $B_Y$

- If $\Delta$ is a $\mathbb{Q}$ divisor then $B_Y$ is a $\mathbb{Q}$ div
- If $G$ is $\mathbb{Q}$-Cartier on $Y$ then
  $f: (X, \Delta + f^* G) \to Y$ is lc-trivial
  and has discriminant $B_Y + G$
  and moduli part $M_Y$.
- If Supp $\Delta^-$ contains no $f^{-1} P$ "generically over $P$"
  for $P \subseteq Y$ of codim 1 then $\gamma_P \leq 1$ and $B_Y \geq 0$

CONSTRUCTION 1

$f: (X, \Delta) \to Y$ lc-trivial $\quad \pi: X' \to X$ birational

$\Delta' = \pi^*(K_X + \Delta) - K_{X'}$.

Then $f' = f \circ \pi : (X', \Delta') \longrightarrow Y$ is an lc-trivial fibration and has same $B_Y$ and $M_Y$ as $f$.

$\rightarrow$ even if $\Delta \geq 0$, $\Delta' \not\geq 0$ in general.

EXAMPLES

(1) Let $f: X \to C$ be relatively minimal elliptic fibration ($X$ smooth surf, $C$ smooth curve)

$g$ fibre all curve, $K_X = f^* D$ $\quad$ $D$ on $C$

$f: (X, 0) \to C$ is a klt-trivial fibration.

[Kodaira] classification of sing fibres and definition of the discriminant

[Kodaira 63, Ueno 73] $K_X \sim f^*(K_C + B_C + M_C)$

$12 M_C \sim j^* \mathcal{O}_{\mathbb{P}^1}(1)$ $\quad$ $j = j$-invariant. $\quad$ $X$ smooth

(2) (Computation of $B_Y$) $f: (X, 0) \longrightarrow C$ lc trivial

$P \in C$ $\quad$ $f^* P = mF$ $\quad$ $F$ smooth

then $\gamma_P = \frac{1}{m}$.

(3) $X = \mathbb{P}^1 \times \mathbb{P}^1$ $\quad$ $D$ reduced divisor on $X$ of bidegree $(d, k)$ with $d \geq 2$. $\Delta = \frac{2}{d} D$.

Then $f$ = second projection $: (X, \Delta) \longrightarrow \mathbb{P}^1$ is lc-trivial (if $d \geq 3$ is klt-trivial)

bidegree of $K_X + \Delta = (-2 + \frac{2}{d} \cdot d, -2 + \frac{2}{d} \cdot k)$

$$= (0, -2 + \tfrac{2}{d} \cdot k)$$

CONSTRUCTION 2

$f : (X, \Delta) \longrightarrow Y$ lc-trivial fibration

$\rho : Y' \longrightarrow Y$ g finite, proper.

$$\begin{tikzcd} (X,\Delta) \arrow[d, "f"'] & (X',\Delta') \arrow[l, "\tau"'] \arrow[d, "f'"] \\ Y & Y' \arrow[l, "\rho"] \end{tikzcd}$$

$X' = \mathrm{norm}(X \times_Y Y')^{\mathrm{main}}$

$\Delta' = \tau^*(K_X + \Delta) - K_{X'}$

$f' : (X', \Delta') \longrightarrow Y'$ is lc-trivial

($f'$ klt trivial if $f$ is)

(*2) in the definition:

$K_X + \Delta \sim_{\mathbb{Q}} f^* D$  $D$ $\mathbb{Q}$-Cartier

$K_{X'} + \Delta' = \tau^*(K_X + \Delta) \sim_{\mathbb{Q}} \tau^* f^* D$

$\| \; (f')^* \rho^* D$

$\mathbb{Q}$ Cart on $Y'$

If $\rho$ birational then

$\rho_* B_{Y'} = B_Y$  $\rho_* M_{Y'} = M_Y$

$\rho_* K_{Y'} = K_Y$

$\Rightarrow \{B_Y\}_Y$ and $\underbrace{\{M_Y\}_Y}_{\mathbb{M}}$ are b-divisors.

Thm (BASE CHANGE PROPERTY) $f : (X, \Delta) \to Y$ lc trivial then $\exists\, \nu : Y' \to Y$ birat sth

(1) $M_{Y'}$ is $\mathbb{Q}$ Cartier and nef

(2) $\forall \mu : Y'' \to Y'$ birat $\mathbb{M}_{Y''} = \mu^* \mathbb{M}_{Y'}$

"$\mathbb{M}$ descends on $Y'$"

$Y'$ is called an Ambro model

Authors of the thm

Ambro ~ 2002 for klt-trivial fibration

Kollár / Fujino-Gongyo for lc-trivial

↖ description of $Y'$

$Y'$ is a model when the singular locus of $f'$ is SNC.

Thm (Inversion of adjunction (Ambro)) $f : (X, \Delta) \to Y$ lc-trivial

$Y$ Ambro model. Then

$(Y, B_Y)$ is klt/lc around $y \in Y$ $\Longleftrightarrow$

$(X, \Delta)$ is klt/lc around $f^{-1}(y) \subseteq X$.

NOTICE A priori $B_Y$ computes only singularities in codimension 1.

« By the thm on $Y$ we extracted all the significant divisors »

CONSTRUCTION 3 $f : (X, \Delta) \to Y$ lc trivial

$(X, \Delta)$ dlt. Let $W \subseteq X$ log canonical centre of $(X, \Delta)$ s.th. $f(W) = Y$.

Then $\exists\, \Delta_W$ on $W$ with $K_X + \Delta|_W = K_W + \Delta_W$

Let $f|_W : W \xrightarrow{g} Y' \to Y$ be the Stein factorisation

One can prove that $g : (W, \Delta_W) \longrightarrow Y'$ is an lctrivial fibration

if $W$ is minimal (among horizontal centres) then $g$ is klt trivial.

Thm (Ambro 2005) $f : (X, \Delta) \to Y$ klt-trivial s.th. $\Delta$ effective over the generic point of $Y$

Then there is a diagram

$$\begin{array}{ccccc} (X,\Delta) & & & & (X^+,\Delta^+) \\ \downarrow f & & & & \downarrow f^+ \\ Y & \xleftarrow{\tau} & \hat{Y} & \xrightarrow{\rho} & Y^+ \end{array}$$

s.th.

1) $\tau$ gen finite, $\rho$ fibration
2) $f^+$ is klt-trivial
3) $(X, \Delta) \times_Y \hat{Y}$ and $(X^+, \Delta^+) \times_{Y^+} \hat{Y}$ are isomorphic over $U \subseteq \hat{Y}$ ($U$ described explicitely)
4) $\tau^* M_Y = \rho^* M_{Y^+}$ and $M_{Y^+}$ is big

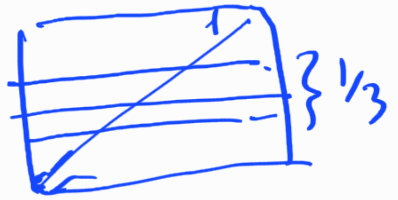
Remark ① If $M_Y \equiv 0$ then $Y^+ = \{p\}$ and, after $\tau$ of finite, $f$ is a product

② $M_Y$ is nef and abundant

$M_Y$ = pback of a big divisor

true also for lc-trivial fibrations [Fujino Gongyo] using constr 3

But the existence of the diagram is <u>not</u> true <u>Ex</u> $X = \mathbb{P}^1 \times \mathbb{P}^1$ $\Delta = \text{diagonal} + \frac{1}{3}([0]+[\infty]+[1])\times\mathbb{P}^1$



Computations: $M_{\mathbb{P}^1} = 0$

3) $\Delta \geq 0$ over the g pt of $Y$ because Hodge th.

Cor $(X,\Delta)$ klt $f:(X,\Delta)\to Y$ klt. $Y$ Ambro model

Then $\exists\, \Delta_Y$ $\mathbb{Q}$ div on $Y$ s.th

$(Y,\Delta_Y)$ klt and $K_X+\Delta \sim_{\mathbb{Q}} f^*(K_Y+\Delta_Y)$

proof

$$K_X+\Delta \sim_{\mathbb{Q}} f^*(\underbrace{K_Y+B_Y}+M_Y)$$

$(Y,B_Y)$ is klt

$\exists\, M \sim_{\mathbb{Q}} M_Y$ $M \geq 0$ with "very small coeff"

$\Rightarrow (Y, B_Y+M)$ is klt. $\square$

For the same result but for $(X,\Delta)$ lc we need $M_Y$ semiample.

<u>B-Semiampleness Conjecture</u> $f:(X,\Delta)\to Y$ lctr

Then $\exists\, \nu: Y'\to Y$ birat s.th $M_{Y'}$ semiample

<u>Effective b-Semiampleness</u> (Prokhorov–Shokurov)

$\exists\, m = m(d,r) \in \mathbb{N}_{>0}$ s.th

$= m$ (invariant of the fiber)

$\forall\, f:(X,\Delta)\to Y$ lc trival $\dim X - \dim Y = d$

and Cartier index of $(K_X+\Delta)|_F = r$

$\exists\ \nu: Y' \to Y$ birational sth $mM_{Y'}$ bpf.

E b Semi is true if
the fibre is $\mathbb{P}^1$ (Pr.-Sh) –
Ell curve (Kod - Ueno) –
S surface w $K_S \sim_{\mathbb{Q}} 0$ Fujino

b Semi the fibre is $S \not\cong \mathbb{P}^2$ (Filipazzi)
if the base is a curve
if $M_Y \equiv 0$ (by Ambro for klt-trivial
Floris for lc-trivial)

Th $\dim Y = 1$ then either $M_Y \equiv 0 \longrightarrow M_Y \sim_{\mathbb{Q}} 0$
or $M_Y$ ample ?? m sth
$mM_Y$ very ample